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NON-LINEAR RESONANCES IN THE FORCED RESPONSES OF PLATES, PART II: ASYMMETRIC RESPONSES OF CIRCULAR PLATES

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The dynamic analogue of the von Karman equations is used to study the forced response, including asymmetric vibrations and traveling waves, of a clamped circular plate subjected to harmonic excitations when the frequency of excitation is near one of the natural frequencies. The method of multiple scales, a perturbation technique, is used to solve the non-linear governing equations. The approach presented provides a great deal of insight into the nature of the non-linear forced resonant response. It is shown that in the absence of internal resonance (i.e., a combination of commensurable natural frequencies) or when the frequency of excitation is near one of the lower frequencies involved in the internal resonance, the steady state response can only have the form of a standing wave. However, when the frequency of excitation is near the highest frequency involved in the internal resonance it is possible for a traveling wave component of the highest mode to appear in the steady state response.

1. INTRODUCTION

It is well known that the large amplitude oscillations of a circular plate can include a traveling wave component [1]. The governing equations are the dynamic analogue of the von Karman equations, which take into account the stretching of the mid-plane. Tobias and Arnold [1], Williams and Tobias [2] and Williams [3] studied the vibrations of so called imperfect disks which exhibit the phenomenon of preferential modes: that is, corresponding to each asymmetric mode of a perfect disk there are two modes having slightly differing frequencies in the imperfect disk. The existence of the traveling wave component in the response was attributed (and confirmed by experiments) to the non-linear coupling between preferential modes. Efstathiades [4] used the Galerkin procedure to analyze the large deflection vibrations of imperfect circular disks. Non-linear vibrations of spinning membrane disks were studied by Advani and Bulkeley [5] and Bulkeley [6] who noted the possibility of traveling waves in the response.

The purpose of this paper is to present a systematic analysis of the forced response, including asymmetric vibrations and traveling waves, of a clamped perfect circular plate subjected to harmonic excitations. The analysis is essentially a generalization of that presented by the authors in an earlier paper [7]. Attention is focused on the response when the frequency of excitation is near one of the natural frequencies. The effects of an internal resonance (combination of commensurable frequencies) in the system are evaluated. The method of multiple scales [8], a perturbation technique, is used in the analysis.

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2. PROBLEM FORMULATION

The equations governing the free, undamped oscillations of non-uniform circular plates were derived by Efstathiades [4]. These equations are simplified to fit the special case of uniform plates, and damping and forcing terms are added. The result is

$$\rho h\, \partial^2 w/\partial t^2 + D\nabla^4 w = L_1(w, F) - c\, \partial w/\partial t + p(r, \theta, t), \tag{1a}$$

$$\nabla^4 F = EhL_2(w), \tag{1b}$$

where

$$L_1(w, F) = \frac{\partial^2 w}{\partial r^2}\left(\frac{1}{r}\frac{\partial F}{\partial r} + \frac{1}{r}\frac{\partial^2 F}{\partial \theta^2}\right) + \frac{\partial^2 F}{\partial r^2}\left(\frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2}\right) - 2\left(\frac{1}{r}\frac{\partial^2 w}{\partial r\,\partial \theta} - \frac{1}{r^2}\frac{\partial F}{\partial \theta}\right)\left(\frac{1}{r}\frac{\partial^2 w}{\partial r\,\partial \theta} - \frac{1}{r^2}\frac{\partial w}{\partial \theta}\right),$$

$$L_2(w) = \left(\frac{1}{r}\frac{\partial^2 w}{\partial r\,\partial \theta} - \frac{1}{r^2}\frac{\partial w}{\partial \theta}\right)^2 - \frac{\partial^2 w}{\partial r^2}\left(\frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2}\right),$$

ρ is the density, h is the thickness, $D = Eh^3/[12(1 - \nu^2)]$, c is the damping coefficient, p is the forcing function, E is Young's modulus, ν is Poisson's ratio, w is the deflection of the middle surface, F is the force function which satisfies the in-plane equilibrium conditions (in-plane inertia is neglected), and

$$\nabla^4 \equiv \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)^2.$$

The relationships between F, w and the in-plane displacements, u_r and u_θ, are given by

$$e_r = \frac{1}{Eh}\left(\frac{1}{r}\frac{\partial F}{\partial r} + \frac{1}{r^2}\frac{\partial^2 F}{\partial \theta^2} - \nu\frac{\partial^2 F}{\partial r^2}\right), \qquad e_\theta = \frac{1}{Eh}\left[\frac{\partial^2 F}{\partial r^2} - \nu\left(\frac{1}{r}\frac{\partial F}{\partial r} + \frac{1}{r^2}\frac{\partial^2 F}{\partial \theta^2}\right)\right],$$

$$e_{r\theta} = \frac{2(1+\nu)}{Eh}\left[\frac{1}{r^2}\frac{\partial F}{\partial \theta} - \frac{1}{r}\frac{\partial^2 F}{\partial r\,\partial \theta}\right], \tag{2a–c}$$

where

$$e_r = \frac{\partial u_r}{\partial r} + \frac{1}{2}\left(\frac{\partial w}{\partial r}\right)^2, \qquad e_\theta = \frac{u_r}{r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{1}{2r^2}\left(\frac{\partial w}{\partial \theta}\right)^2, \qquad e_{r\theta} = \frac{1}{r}\frac{\partial u_r}{\partial r} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} + \frac{1}{r}\frac{\partial w}{\partial r}\frac{\partial w}{\partial \theta}. \tag{3a–c}$$

It is convenient to rewrite these equations in terms of dimensionless variables, denoted by overbars, which are defined as follows:

$$r = a\bar{r}, \qquad t = a^2(\rho h/D)^{1/2}\,\bar{t}, \qquad w = (h^2/a)\,\bar{w}, \qquad (u_r, u_\theta) = (h^4/a^3)(\bar{u}_r, \bar{u}_\theta),$$

$$c = [24(1-\nu^2)/a^4](\rho h^5 D)^{1/2}\,\bar{c}, \qquad p = [12(1-\nu^2)\,Dh^4/a^7]\,\bar{p}, \qquad F = (Eh^5/a^2)\,\bar{F},$$

where a is the radius of the plate. We are concerned with generating an approximate solution which is valid as h/a approaches zero; each of the dimensionless variables defined above is presumed to be $O(1)$ in this limit. In addition, we define $\bar{e}_r$, $\bar{e}_\theta$ and $\bar{e}_{r\theta}$, which are also presumed to be $O(1)$ as h/a approaches zero, as follows:

$$(e_r, e_\theta, e_{r\theta}) = (h^4/a^4)(\bar{e}_r, \bar{e}_\theta, \bar{e}_{r\theta}).$$

Substituting these definitions into equations (1) and (2) and dropping the overbars in the result, one obtains

$$\partial^2 w/\partial t^2 + \nabla^4 w = \varepsilon[L_1(w_1 F) - 2c\,\partial w/\partial t + p], \qquad \nabla^4 F = L_2(w), \tag{4a, b}$$

where $\varepsilon = 12(1-\nu^2)h^2/a^2$,

$$e_r = \frac{1}{r}\frac{\partial F}{\partial r} + \frac{1}{r^2}\frac{\partial^2 F}{\partial \theta^2} - \nu\frac{\partial^2 F}{\partial r^2}, \qquad e_\theta = \frac{\partial^2 F}{\partial r^2} - \nu\left(\frac{1}{r}\frac{\partial F}{\partial r} + \frac{1}{r^2}\frac{\partial^2 F}{\partial \theta^2}\right),$$

$$e_{r\theta} = 2(1+\nu)\left(\frac{1}{r^2}\frac{\partial F}{\partial \theta} - \frac{1}{r}\frac{\partial^2 F}{\partial r\,\partial \theta}\right), \tag{5a–c}$$

and the form of equations (3) is not changed.

The boundary conditions are developed for plates which are clamped along a circular edge. For all t, and θ,

$$w = 0, \qquad \partial w/\partial r = 0, \qquad \text{at} \qquad r = a, \tag{6a}$$

$$u_r = 0, \qquad u_\theta = 0, \qquad \text{at} \qquad r = a. \tag{6b}$$

It follows from equations (3), (5) and (6) that, for all t and θ,

$$e_\theta = 0, \qquad \frac{\partial}{\partial r}(re_\theta) - e_r - \frac{\partial}{\partial \theta}(e_{r\theta}) = 0, \qquad \text{at} \qquad r = a,$$

$$\frac{\partial^2 F}{\partial r^2} - \nu\left(\frac{1}{r}\frac{\partial F}{\partial r} + \frac{1}{r^2}\frac{\partial^2 F}{\partial \theta^2}\right) = 0, \qquad \text{at } r = a, \tag{7a}$$

$$\frac{\partial^3 F}{\partial r^2} + \frac{1}{r}\frac{\partial^2 F}{\partial r^2} - \frac{1}{r^2}\frac{\partial F}{\partial r} + \frac{(2+\nu)}{r^2}\frac{\partial^2 F}{\partial r\,\partial \theta^2} - \frac{(3+\nu)}{r^3}\frac{\partial^2 F}{\partial \theta^2} = 0, \qquad \text{at} \qquad r = a. \tag{7b}$$

In addition, it is necessary to require the solution to be bounded at $r = 0$.

It is noted that, when ε is small, w is much smaller than h. Had w been the same order as h (say, $w = h\bar{w}$), then no small parameter would have appeared in equation (4a) and the linear and non-linear terms would have been the same order. Hence, the present approach must be viewed as one which provides corrections for the small-deflection theory (for which w is much smaller than h) and not as one which provides a solution for the large-deflection theory (for which w is the same order as h). This means that some typical non-linear phenomena such as jump phenomena, modal interactions, etc., can be part of the corrected small-deflection theory.

Further, it is noted that w is a function of r, θ and t and the solution may contain traveling waves. Equations (4) do not lend themselves to a straightforward separation of the spatial and temporal variables. However, by using the method of multiple scales, an asymptotic expansion of the solution of equations (4) can still be constructed. The expansion is to be uniformly valid for small ε and large t.

3. SOLUTION

Following the derivative-expansion version of the method of multiple scales (see reference [7]), we expand w and F as follows:

$$w(r,\theta,t;\varepsilon) \sim \sum_{j=0}^{\infty} \varepsilon^j w_j(r,\theta,T_0,T_1,\ldots), \qquad F(r,\theta,t;\varepsilon) \sim \sum_{j=0}^{\infty} \varepsilon^j F_j(r,\theta,T_0,T_1,\ldots) \tag{8a, b}$$

where $T_n = \varepsilon^n t$.

Substituting equations (8) into equations (4) and equating coefficients of like powers of ε yields

$$D_0^2 w_0 + \nabla^4 w_0 = 0, \tag{9}$$

$$\nabla^4 F_0 = \left(\frac{1}{r}\frac{\partial^2 w_0}{\partial r\,\partial\theta} - \frac{1}{r^2}\frac{\partial w_0}{\partial\theta}\right)^2 - \frac{\partial^2 w_0}{\partial r^2}\left(\frac{1}{r}\frac{\partial w_0}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w_0}{\partial\theta^2}\right), \tag{10}$$

$$D_0^2 w_1 + \nabla^4 w_1 = -2D_0 D_1 w_0 - 2cD_0 w_0 + p + \frac{\partial^2 w_0}{\partial r^2}\left(\frac{1}{r}\frac{\partial F_0}{\partial r} + \frac{1}{r^2}\frac{\partial^2 F_0}{\partial\theta^2}\right)$$
$$+ \frac{\partial^2 F_0}{\partial r^2}\left(\frac{1}{r}\frac{\partial w_0}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w_0}{\partial\theta^2}\right) - 2\left(\frac{1}{r}\frac{\partial^2 F_0}{\partial r\,\partial\theta} - \frac{1}{r^2}\frac{\partial F_0}{\partial\theta}\right)\left(\frac{1}{r}\frac{\partial^2 w_0}{\partial r\,\partial\theta} - \frac{1}{r^2}\frac{\partial w_0}{\partial\theta}\right), \tag{11}$$

etc., where $D_n = \partial/\partial T_n$.

Substituting equations (8) into equations (6a) and (7) and equating coefficients of like powers of ε, one obtains (letting $a = 1$)

$$w_j = 0, \qquad \partial w_j/\partial r = 0, \tag{12a, b}$$

$$\partial^2 F_j/\partial r^2 - \nu(\partial F_j/\partial r + \partial^2 F_j/\partial\theta^2) = 0, \tag{13a}$$

$$\partial^3 F_j/\partial r^3 + \partial^2 F_j/\partial r^2 - \partial F_j/\partial r + (2+\nu)\,\partial^3 F_j/\partial r\,\partial\theta^2 - (3+\nu)\,\partial^2 F_j/\partial\theta^2 = 0, \tag{13b}$$

for all j, θ and t at $r = 1$. In addition, it is necessary to require w_j and F_j, for all j, to be bounded at $r = 0$.

It follows from equations (9) and (12) that

$$w_0 = \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} \phi_{nm}(r)\{A_{nm}\exp[\mathrm{i}(\omega_{nm}T_0 + n\theta)] + B_{nm}\exp[\mathrm{i}(\omega_{nm}T_0 - n\theta)] + \mathrm{cc}\} \tag{14}$$

where the $\phi_{nm}(r)$ are the linear, free oscillation modes given by

$$\phi_{nm}(r) = \kappa_{nm}[\mathrm{J}_n(\eta_{nm}r) - \{\mathrm{J}_n(\eta_{nm})/\mathrm{I}_n(\eta_{nm})\}\,\mathrm{I}_n(\eta_{nm}r)],$$

the κ_{nm} are chosen so that

$$\int_0^1 r\phi_{nm}^2(r)\,\mathrm{d}r = 1,$$

the η_{nm} are the roots of $\mathrm{I}_n(\eta)\,\mathrm{J}_n'(\eta) - \mathrm{I}_n'(\eta)\,\mathrm{J}_n(\eta) = 0$, $\omega_{nm} = \eta_{nm}^2$, the A_{nm} and the B_{nm} are complex functions of all the T_n for $n \geqslant 1$ which are to be determined from the solvability conditions at the next level of approximation, and cc represents the complex conjugate of the preceding terms.

It is noted that the solution given by equation (14) contains both traveling and standing waves depending on the relative values of the A_{nm} and B_{nm}. The solution can also be written in the following equivalent form:

$$w_0 = \sum_{n=-\infty}^{\infty}\sum_{m=1}^{\infty} \phi_{nm}(r)\,u_{nm}(T_0, T_1, \ldots)\exp(\mathrm{i}n\theta), \tag{15}$$

where

$$u_{nm} = A_{nm}\exp(\mathrm{i}\omega_{nm}T_0) + \bar{B}_{nm}\exp(-\mathrm{i}\omega_{nm}T_0), \tag{16}$$

$\phi_{-nm} = \phi_{nm}$ and $\omega_{-nm} = \omega_{nm}$. Because w_0 is real,

$$A_{-nm} = B_{nm} \qquad \text{and} \qquad B_{-nm} = A_{nm}. \tag{17}$$

Substituting equation (15) into equation (10) leads to

$$\nabla^4 F_0 = \sum_{n,\, p=-\infty}^{\infty} \sum_{m,\, q=1}^{\infty} E(nm, pq)\, u_{nm} u_{pq} \exp[\mathrm{i}(n+p)\theta] \tag{18}$$

where

$$E(nm, pq) = \frac{-np}{r^2}\left(\phi'_{nm} - \frac{\phi_{nm}}{r}\right)\left(\phi'_{pq} - \frac{\phi_{pq}}{r}\right) - \frac{1}{2r}(\phi'_{nm}\phi'_{pq})' + \frac{1}{2r^2}(p^2\phi''_{nm}\phi_{pq} + n^2\phi''_{pq}\phi_{nm})$$

and primes denote differentiation with respect to r.

An expansion for F_0 is assumed in the following form:

$$F_0 = \sum_{n=-\infty}^{\infty} U_n(r, T_0, T_1, \ldots) \exp(\mathrm{i}n\theta). \tag{19}$$

Substituting equation (19) into equation (18), multiplying the result by $\exp(-\mathrm{i}a\theta)$, and integrating from $\theta = 0$ to $\theta = 2\pi$, we obtain

$$\nabla_a^4 U_a = \sum_{n=-\infty}^{\infty} \sum_{m,\, q=1}^{\infty} E(nm, pq)\, u_{nm} u_{pq}, \tag{20}$$

where

$$p = a - n \tag{21}$$

and

$$\nabla_a^4 = [\partial^2/\partial r^2 + (1/r)\,\partial/\partial r - a^2/r^2]^2.$$

Then U_a is further expanded as

$$U_a = \sum_{n=1}^{\infty} v_{an}(T_0, T_1, \ldots)\, \psi_{an}(r), \tag{22}$$

where the ψ_{an} are the eigenfunctions of the following problem:

$$(\nabla_a^4 - \xi_{an}^4)\, \psi_{an} = 0 \qquad \text{in} \qquad r = [0, 1],$$

where ψ_{an} is bounded at $r = 0$ and, from equations (13),

$$\psi''_{an} - \nu(\psi'_{an} - a^2\psi_{an}) = 0 \qquad \text{and} \qquad \psi'''_{an} + \psi''_{an} - \psi'_{an} - a^2[(2+\nu)\psi'_{an} - (3+\nu)\psi_{an}] = 0$$

for all θ and t at $r = 1$. It follows that

$$\psi_{an} = \tilde{\kappa}_{an}[\mathrm{J}_a(\xi_{an} r) - \tilde{c}_{an}\,\mathrm{I}_a(\xi_{an} r)], \tag{23}$$

where the $\tilde{\kappa}_{an}$ are chosen so that

$$\int_0^1 r\psi_{an}^2\, \mathrm{d}r = 1,$$

$$\tilde{c}_{an} = \frac{[a(a+1)(\nu+1) - \xi_{an}^2]\,\mathrm{J}_a(\xi_{an}) - \xi_{an}(\nu+1)\,\mathrm{J}_{a-1}(\xi_{an})}{[a(a+1)(\nu+1) + \xi_{an}^2]\,\mathrm{I}_a(\xi_{an}) - \xi_{an}(\nu+1)\,\mathrm{I}_{a-1}(\xi_{an})},$$

and ξ_{an} are the roots of

$$a^2(a+1)(\nu+1)[\mathrm{J}_a(\xi_{an}) - \tilde{c}_{an}\mathrm{I}_a(\xi_{an})] - a\xi_{an}^2(\nu+1)[\mathrm{J}_{a-1}(\xi_{an}) - \tilde{c}_{an}\mathrm{I}_{a-1}(\xi_{an})]$$
$$+ a\xi_{an}^2[\mathrm{J}_a(\xi_{an}) + \tilde{c}_{an}\mathrm{I}_a(\xi_{an})] - \xi_{an}^3[\mathrm{J}_{a-1}(\xi_{an}) + \tilde{c}_{an}\mathrm{I}_{a-1}(\xi_{an})] = 0.$$

Substituting equation (22) into equation (20), multiplying the result by $r\psi_{ab}$, and then integrating from $r=0$ to $r=1$, one obtains

$$v_{ab}(T_0, T_1, \ldots) = \sum_{n=-\infty}^{\infty} \sum_{m,\,q=1}^{\infty} G(nm, pq; ab)\, u_{nm} u_{pq}, \tag{24}$$

where

$$G(nm, pq; ab) = \xi_{ab}^{-4} \int_0^1 r\psi_{ab} E(nm, pq)\,\mathrm{d}r \tag{25}$$

and p, a and n are related according to equation (21). It follows from equations (24), (22) and (19) that

$$F_0 = \sum_{a,\,n=-\infty}^{\infty} \sum_{b,\,m,\,q=1}^{\infty} \psi_{ab} G(nm, pq; ab)\, u_{nm} u_{pq} \exp(\mathrm{i}a\theta), \tag{26}$$

where $p = a - n$.

Substituting equations (26) and (14) into equation (11) leads to

$$\begin{aligned} D_0^2 w_1 + \nabla^4 w_1 = {} & \sum_{n=-\infty}^{\infty} \sum_{m=1}^{\infty} -2\mathrm{i}\omega_{nm}\phi_{nm} [(D_1 A_{nm} + c_{nm} A_{nm}) \exp(\mathrm{i}\omega_{nm} T_0) - (D_1 \bar{B}_{nm} + c_{nm} \bar{B}_{nm}) \\ & \times \exp(-\mathrm{i}\omega_{nm} T_0)] \exp(\mathrm{i}n\theta) + \left[\sum_{n=-\infty}^{\infty} \sum_{m=1}^{\infty} P_{nm} \phi_{nm} \exp(\mathrm{i}n\theta) \right] \cos \lambda T_0 \\ & + \sum_{a,\,n,\,c=-\infty}^{\infty} \sum_{b,\,m,\,d,\,q=1}^{\infty} G(nm, pq; ab)\, \hat{E}(cd, ab)\, u_{cd} u_{pq} u_{nm} \exp[\mathrm{i}(a+c)\theta], \end{aligned} \tag{27}$$

where modal damping has been assumed, p has been expanded as

$$p(r, \theta, t) = \left[\sum_{n=-\infty}^{\infty} \sum_{m=1}^{\infty} P_{nm} \phi_{nm} \exp(\mathrm{i}n\theta) \right] \cos \lambda T_0$$

and

$$\hat{E}(cd, ab) = \frac{\phi''_{cd}}{r}\left(\psi'_{ab} - \frac{a^2}{r}\psi_{ab}\right) + \frac{\psi''_{ab}}{r}\left(\phi'_{cd} - \frac{c^2}{r}\phi_{cd}\right) + \frac{2ac}{r^2}\left(\psi'_{ab} - \frac{1}{r}\psi_{ab}\right)\left(\phi'_{cd} - \frac{1}{r}\phi_{cd}\right).$$

Because w_1 and w_0 satisfy the same boundary conditions, an expansion for w_1 is assumed in the form

$$w_1 = \sum_{n=-\infty}^{\infty} \sum_{m=1}^{\infty} H_{nm}(T_0, T_1, \ldots)\, \phi_{nm}(r) \exp(\mathrm{i}n\theta). \tag{28}$$

Substituting equation (28) into equation (27), multiplying the result by $r\phi_{kl}(r)\exp(-\mathrm{i}k\theta)$, and integrating the result from $r=0$ to 1 and $\theta=0$ to 2π, one obtains

$$\begin{aligned} D_0^2 H_{kl} + \omega_{kl}^2 H_{kl} = {} & 2\mathrm{i}\omega_{kl}[(D_1 A_{kl} + c_{kl} A_{kl}) \exp(\mathrm{i}\omega_{kl} T_0) - (D_1 \bar{B}_{kl} + c_{kl} \bar{B}_{kl}) \exp(-\mathrm{i}\omega_{kl} T_0)] \\ & + P_{kl}[\exp(\mathrm{i}\lambda T_0) + \exp(-\mathrm{i}\lambda T_0)] + \sum_{p,\,n,\,c=-\infty}^{\infty} \sum_{d,\,m,\,q=1}^{\infty} \Gamma(kl, cd, nm, pq) \\ & \times \left[\sum_{j=1}^{8} S_j \exp(\mathrm{i}\Lambda_j T_0) \right], \qquad k = 1, 2, \ldots, \qquad l = 1, 2, \ldots, \end{aligned} \tag{29}$$

where

$$\Gamma(kl, cd, nm, pq) = \sum_{b=1}^{\infty} G(nm, pq; ab) \int_0^1 r\phi_{kl} \hat{E}(cd, ab)\,\mathrm{d}r, \tag{30a}$$

$$a = k - c, \qquad p = k - c - n, \tag{30b, c}$$

Λ_j are frequency combinations, and S_j are functions of A_{nm} and B_{nm}. Both Λ_j and S_j are listed in the Appendix.

The solvability conditions can be obtained by requiring the coefficients of $\exp(\pm i\omega_{kl}T_0)$ to vanish from the right-hand sides of equation (29). In general the solvability conditions can be written as

$$-2i\omega_{kl}(D_1 A_{kl} + c_{kl} A_{kl}) + A_{kl} \sum_{n=-\infty}^{\infty} \sum_{m=1}^{\infty} \gamma_{klnm}(A_{nm}\bar{A}_{nm} + B_{nm}\bar{B}_{nm}) + N_{kl}^A + R_{kl}^A = 0, \quad (31a)$$

$$2i\omega_{kl}(D_1 \bar{B}_{kl} + c_{kl} \bar{B}_{kl}) + \bar{B}_{kl} \sum_{n=-\infty}^{\infty} \sum_{m=1}^{\infty} \gamma_{klnm}(A_{nm}\bar{A}_{nm} + B_{nm}\bar{B}_{nm}) + N_{k}^B + R_{k}^B = 0, \quad (31b)$$

where $R_{kl}^{A,B}$ are terms due to internal resonances, if any, $N_{kl}^{A,B}$ are terms due to the external excitation, if any, and γ_{klnm} are constants. We made use of equations (17) and (30c) to arrive at the double sums in equations (31).

4. THE CASE OF NO INTERNAL RESONANCE

In the absence of internal resonances $R_{kl}^{A,B} = 0$ in equations (31). When λ is near ω_{rs},

$$\lambda = \omega_{rs} + \varepsilon\sigma, \quad (32)$$

$$N_{rs}^A = \tfrac{1}{2}\hat{P}_{rs}\exp(i\sigma T_1), \qquad N_{rs}^B = \hat{P}_{rs}\exp(-i\sigma T_1) \quad (33a,b)$$

and

$$N_{k}^{A,B} = 0, \qquad \text{for} \qquad kl \neq rs, \quad (33c)$$

where σ is a detuning parameter. Next we let

$$A_{nm} = \tfrac{1}{2}a_{nm}\exp(i\alpha_{nm}) \qquad \text{and} \qquad B_{nm} = \tfrac{1}{2}b_{nm}\exp(i\beta_{nm}) \quad (34a,b)$$

where a_{nm}, b_{nm}, α_{nm} and β_{nm} are real functions of T_1.

Substituting equations (33) and (34) into (31) and separating the result into real and imaginary parts yields

$$\omega_{kl}(a'_{kl} + c_{kl}a_{kl}) = 0, \qquad \omega_{kl}a_{kl}\alpha'_{kl} + \tfrac{1}{8}a_{kl}s_{kl} = 0, \quad (35a,b)$$

$$\omega_{kl}(b'_{kl} + c_{kl}b_{kl}) = 0, \qquad \omega_{kl}b_{kl}\beta'_{kl} + \tfrac{1}{8}b_{kl}s_{kl} = 0, \quad (35c,d)$$

for $kl \neq rs$,

$$\omega_{rs}(a'_{rs} + c_{rs}a_{rs}) - \tfrac{1}{2}P_{rs}\sin\mu_{rs}^a = 0, \qquad \omega_{rs}a_{rs}\alpha'_{rs} + \tfrac{1}{8}a_{rs}s_{rs} + \tfrac{1}{2}P_{rs}\cos\mu_{rs}^a = 0, \quad (36a,b)$$

$$\omega_{rs}(b'_{rs} + c_{rs}b_{rs}) - \tfrac{1}{2}P_{rs}\sin\mu_{rs}^b = 0, \qquad \omega_{rs}b_{rs}\beta'_{rs} + \tfrac{1}{8}b_{rs}s_{rs} + \tfrac{1}{2}P_{rs}\cos\mu_{rs}^b = 0, \quad (36c,d)$$

where

$$s_{kl} = \sum_{n=-\infty}^{\infty} \sum_{m=1}^{\infty} \gamma_{klnm}(a_{nm}^2 + b_{nm}^2),$$

$$\mu_{rs}^a = \sigma T_1 - \alpha_{rs} \qquad \text{and} \qquad \mu_{rs}^b = \sigma T_1 - \beta_{rs}. \quad (37a,b)$$

For the steady state solution, the a_{nm}, b_{nm}, μ_{rs}^a and μ_{rs}^b are constants. It follows immediately from equations (35) that

$$a_{kl} = b_{kl} = 0, \qquad \text{for} \qquad kl \neq rs$$

and from equations (36) that neither a_{rs} nor b_{rs} can be zero. Thus, the steady state solution is given by equations (36) which can be rewritten as

$$\omega_{rs}c_{rs} = (P_{rs}/2a_{rs})\sin\mu_{rs}^a, \quad (38a)$$

$$\omega_{rs}\sigma + \tfrac{1}{8}\gamma_{rsrs}(a_{rs}^2 + b_{rs}^2) = -(P_{rs}/2a_{rs})\cos\mu_{rs}^a, \quad (38b)$$

$$\omega_{rs} c_{rs} = (P_{rs}/2b_{rs}) \sin \mu_{rs}^b, \tag{39a}$$

$$\omega_{rs} \sigma + \tfrac{1}{8}\gamma_{rsrs}(a_{rs}^2 + b_{rs}^2) = -(P_{rs}/2b_{rs}) \cos \mu_{rs}^b. \tag{39b}$$

Squaring and adding equations (38) and comparing the result with that obtained by squaring and adding equations (39), we obtain

$$b_{rs} = a_{rs} \quad \text{and} \quad \mu_{rs}^b = \mu_{rs}^a. \tag{40a, b}$$

Therefore, using equations (37), (34) and (14), one can write the steady state response as

$$w = 2\phi_{rs} a_{rs} \cos(\lambda t - \mu_{rs}^a) \cos r\theta + O(\varepsilon). \tag{41}$$

Consequently, in the absence of internal resonances, the steady state forced response consists of standing waves only. One can describe the response with a single mode having a frequency equal to that of the excitation, as several investigators have done previously, the solution being essentially that of the Duffing equation.

5. EFFECTS OF AN INTERNAL RESONANCE

In this section consideration is given to the effects of an internal resonance involving four modes: that is, combination of commensurable frequencies of the form

$$\omega_{CO} + \omega_{NM} + \omega_{PQ} \approx \omega_{KL}. \tag{42}$$

Further, we assume that these frequencies are such that equation (30c) is satisfied: that is

$$K = C + N + P. \tag{43}$$

(For a clamped circular plate equations (42) and (43) are satisfied by the following natural frequencies (see, e.g., reference [9]): $\omega_{01} = 10{\cdot}22$, $\omega_{02} = 39{\cdot}77$, $\omega_{21} = 34{\cdot}88$ and $\omega_{22} = 84{\cdot}58$; so that $\omega_{01} + \omega_{02} + \omega_{21} = 84{\cdot}87 \approx \omega_{22}$. The first subscript refers to the number of nodal diameters and the second subscript refers to the number of nodal circles including the boundary.) To characterize the approximation in equation (42), we introduce a detuning parameter, σ_1, as follows:

$$\omega_{CD} + \omega_{NM} + \omega_{PQ} + \varepsilon\sigma_1 = \omega_{KL}. \tag{44}$$

The terms due to the internal resonance, $R_{kl}^{A,B}$, appearing in the solvability conditions (31), which can be obtained by considering the Appendix and equations (17), (43) and (44), are

$$\begin{aligned}
R_{KL}^A &= Q_{KL}(A_{CD} A_{NM} A_{PQ} + B_{CD} B_{NM} B_{PQ}) \exp(-i\sigma_1 T_1),\\
R_{PQ}^A &= Q_{PQ}(A_{KL} \bar{A}_{CD} \bar{A}_{NM} + B_{KL} \bar{B}_{CD} \bar{B}_{NM}) \exp(i\sigma_1 T_1),\\
R_{NM}^A &= Q_{NM}(A_{KL} \bar{A}_{PQ} \bar{A}_{CD} + B_{KL} \bar{B}_{PQ} \bar{B}_{CD}) \exp(i\sigma_1 T_1),\\
R_{CD}^A &= Q_{CD}(A_{KL} \bar{A}_{NM} \bar{A}_{PQ} + B_{KL} \bar{B}_{NM} \bar{B}_{PQ}) \exp(i\sigma_1 T_1)\\
\text{and} \quad R_{kl}^A &= 0 \quad \text{for} \quad kl \neq KL, PQ, NM, CD,
\end{aligned}$$

where the Q's are constants. The expressions for R_{kl}^B can be obtained from those of R_{kl}^A by replacing A_{kl} by $\bar{B}_{kl}$, B_{kl} by $\bar{A}_{kl}$ and σ_1 by $-\sigma_1$.

Substituting equations (34) and the expressions for $R_{kl}^{A,B}$ and $N_{kl}^{A,B}$ into equations (31) and separating the result into real and imaginary parts leads to the following solvability conditions:

$$\begin{aligned}
&\omega_{kl}(a_{kl}' + c_{kl} a_{kl}) - \tfrac{1}{8}Q_{kl} S_{kl}^1 - N_{kl} \sin \mu_{kl}^a = 0,\\
&\omega_{kl}(b_{kl}' + c_{kl} b_{kl}) - \tfrac{1}{8}Q_{kl} S_{kl}^1 - N_{kl} \sin \mu_{kl}^b = 0,
\end{aligned} \tag{45a, b}$$

$$\begin{aligned}
&\omega_{kl} a_{kl} \alpha_k' + \tfrac{1}{8}a_{kl} s_{kl} + \tfrac{1}{8}Q_{kl} S_{kl}^2 + N_{kl} \cos \mu_{kl}^a = 0,\\
&\omega_{kl} b_{kl} \beta_{kl}' + \tfrac{1}{8}b_{kl} s_{kl} + \tfrac{1}{8}Q_{kl} S_{kl}^2 + N_{kl} \cos \beta_{kl}^b = 0,
\end{aligned} \tag{46a, b}$$

for $kl = CD, NM, PQ$,

$$\omega_{KL}(a'_{KL} + c_{KL}a_{KL}) + \tfrac{1}{8}Q_{KL}S^1_{KL} - N_{KL}\sin\mu^a_{KL} = 0,$$
$$\omega_{KL}(b'_{KL} + c_{KL}b_{KL}) + \tfrac{1}{8}Q_{KL}S^1_{KL} - N_{KL}\sin\mu^b_{KL} = 0, \quad (47a, b)$$

$$\omega_{KL}a_{KL}\alpha'_{KL} + \tfrac{1}{8}a_{KL}s_{KL} + \tfrac{1}{8}Q_{KL}S^2_{KL} - N_{KL}\cos\mu^a_{KL} = 0,$$
$$\omega_{KL}b_{KL}\beta'_{KL} + \tfrac{1}{8}b_{KL}s_{KL} + \tfrac{1}{8}Q_{KL}S^2_{KL} + N_{KL}\cos\mu^b_{KL} = 0, \quad (48a, b)$$

for $kl = KL$ and

$$\omega_{kl} + (a'_{kl} + c_{kl}a_{kl}) - N_{kl}\sin\mu^a_{kl} = 0, \qquad \omega_{kl}(b'_{kl} + c_{kl}b_{kl}) - N_{kl}\sin\mu^b_{kl} = 0, \quad (49a, b)$$

$$\omega_{kl}a_{kl}\alpha'_{kl} + \tfrac{1}{8}a_{kl}s_{kl} + N_{kl}\cos\mu^a_{kl} = 0, \qquad \omega_{kl}b_{kl}\beta'_{kl} + \tfrac{1}{8}b_{kl}s_{kl} + N_{kl}\cos\mu^b_{kl} = 0, \quad (50a, b)$$

for $kl \neq CD, NM, PQ$ and KL, where

$$s_{kl} = \sum_{n=-\infty}^{\infty}\sum_{m=1}^{\infty}\gamma_{klnm}(a^2_{nm} + b^2_{nm}),$$

$$N_{kl} = \tfrac{1}{2}P_{kl} \text{ when } \lambda \text{ is near } \omega_{kl}, \qquad N_{kl} = 0 \text{ when } \lambda \text{ is away from } \omega_{kl}, \quad (51a, b)$$

$$\mu^a_{kl} = \sigma_2 T_1 - \alpha_{kl}, \qquad \mu^b_{kl} = \sigma_2 T_1 - \beta_{kl}, \quad (52a, b)$$

the detuning parameter σ_2 being defined by $\lambda = \omega_{kl} + \varepsilon\sigma_2$ when λ is near ω_{kl},

$$S^1_{CD} = a_{NM}a_{PQ}a_{KL}\sin\tilde{\mu}_A + b_{NM}b_{PQ}b_{KL}\sin\tilde{\mu}_B,$$
$$S^2_{CD} = a_{NM}a_{PQ}a_{KL}\cos\tilde{\mu}_A + b_{NM}b_{PQ}b_{KL}\cos\tilde{\mu}_B,$$
$$S^1_{NM} = a_{PQ}a_{KL}a_{CD}\cos\tilde{\mu}_A + b_{PQ}b_{KL}b_{CD}\cos\tilde{\mu}_B,$$
$$S^2_{NM} = a_{PQ}a_{KL}a_{CD}\cos\tilde{\mu}_A + b_{PQ}b_{KL}b_{CD}\cos\tilde{\mu}_B,$$
$$S^1_{PQ} = a_{KL}a_{CD}a_{NM}\sin\tilde{\mu}_A\, b_{KL}b_{CD}b_{NM}\sin\tilde{\mu}_B,$$
$$S^2_{PQ} = a_{KL}a_{CD}a_{NM}\cos\tilde{\mu}_A + b_{KL}b_{CD}b_{NM}\cos\tilde{\mu}_B,$$
$$S^1_{KL} = a_{CD}a_{NM}a_{PQ}\sin\tilde{\mu}_A + b_{CD}b_{NM}b_{PQ}\sin\tilde{\mu}_B,$$
$$S^2_{KL} = a_{CD}a_{NM}a_{PQ}\cos\tilde{\mu}_A + b_{CD}b_{NM}b_{PQ}\cos\tilde{\mu}_B,$$

$$\tilde{\mu}_A = \sigma_1 T_1 - \alpha_{CD} - \alpha_{NM} - \alpha_{PQ} + \alpha_{KL}, \qquad \tilde{\mu}_B = \sigma_1 T_1 - \beta_{CD} - \beta_{NM} - \beta_{PQ} + \beta_{KL}.$$

It is noted that equations (45)–(50) are analogous to equations (29)–(32) of reference [7]. For a steady state solution, all a_{kl}, b_{kl}, $\tilde{\mu}_A$, $\tilde{\mu}_B$, μ^a_{kl} and μ^b_{kl} are constants. This leads to

$$\omega_{kl}c_{kl}a_{kl} - \tfrac{1}{8}Q_{kl}S^1_{kl} - N_{kl}\sin\mu^a_{kl} = 0, \qquad \omega_{kl}c_{kl}b_{kl} - \tfrac{1}{8}Q_{kl}S^1_{kl} - N_{kl}\sin\mu^b_{kl} = 0, \quad (53a, b)$$

for $kl = CD$, NM, and PQ,

$$\omega_{KL}c_{KL}a_{KL} + \tfrac{1}{8}Q_{KL}S^1_{KL} - N_{KL}\sin\mu^a_{KL} = 0, \qquad \omega_{KL}c_{KL}b_{KL} + \tfrac{1}{8}Q_{KL}S^1_{KL} - N_{KL}\sin\mu^b_{KL} = 0, \quad (54a, b)$$

for $kl = KL$, and

$$\omega_{kl}c_{kl}a_{kl} - N_{kl}\sin\mu^a_{kl} = 0, \qquad \omega_{kl}c_{kl}b_{kl} - N_{kl}\cos\mu^b_{kl} = 0, \quad (55a, b)$$

for $kl \neq CD$, NM, PQ and KL,

$$\tilde{\mu}'_A = \sigma_1 - \alpha'_{CD} - \alpha'_{NM} - \alpha'_{PQ} + \alpha'_{KL} = 0, \qquad \tilde{\mu}'_B = \sigma_1 - \beta'_{CD} - \beta'_{NM} - \beta'_{PQ} + \beta'_{KL} = 0, \quad (56a, b)$$

where α'_{kl} and β'_{kl} are given by equations (46) for $kl = CD$, NM and PQ and by equations (48) for $kl = KL$,

$$\mu^{a\prime}_{rs} = \sigma_2 - \alpha'_{rs} = 0, \qquad \mu^{b\prime}_{rs} = \sigma_2 - \beta'_{rs} = 0, \quad (57a, b)$$

when λ is near ω_{rs}. Several possibilities are considered next.

5.1. THE CASE OF λ AWAY FROM ALL ω_{kl}

In this case $N_{kl} = 0$ for all kl. Thus equations (53) and (54) lead to

$$a_{kl} = b_{kl}, \qquad \text{for} \qquad kl = CD, NM, PQ \quad \text{and} \quad KL$$

and equations (55) lead to

$$a_{kl} = b_{kl} = 0, \qquad \text{for} \qquad kl \neq CD, NM, PQ \quad \text{and} \quad KL.$$

Assuming non-trivial solutions for a_{CD}, a_{NM}, a_{PQ}, and a_{KL}, one finds from equations (53) and (54) that

$$(a_{CD}/a_{KL})^2 = -\omega_{KL} c_{KL} Q_{CD}/\omega_{CD} c_{CD} Q_{KL}, \tag{58a}$$

$$(a_{NM}/a_{KL})^2 = -\omega_{KL} c_{KL} Q_{NM}/\omega_{NM} c_{NM} Q_{KL}, \tag{58b}$$

$$(a_{PQ}/a_{KL})^2 = -\omega_{KL} c_{KL} Q_{PQ}/\omega_{PQ} c_{PQ} Q_{KL}. \tag{58c}$$

However, for mechanical systems and structural elements, non-trivial solutions cannot exist in the absence of external excitations and in the presence of linear viscous damping: that is, the systems cannot be self-excited. Consequently, the signs of Q_{CD}, Q_{NM}, Q_{PQ} and Q_{KL} must be the same so that the relationships given by equations (58) are impossible and thus

$$a_{CD} = a_{NM} = a_{PQ} = a_{KL} = 0.$$

5.2. THE CASE OF λ NEAR ω_{kl}, $kl \neq CD, NM, PQ$ AND KL

In this case $N_{CD} = N_{NM} = N_{PQ} = N_{KL} = 0$, and it follows that $a_{CD} = a_{NM} = a_{PQ} = a_{KL} = 0$ in the steady-state solution, which is governed by equations (49) and (50). These equations are identical in structure to equations (36). Hence, the steady-state response is a standing wave of the form

$$w = 2\phi_{kl} a_{kl} \cos(\lambda t - \mu_{kl}^a) \cos k\theta + O(\varepsilon). \tag{59}$$

5.3. THE CASE OF λ NEAR ω_{CD}

In this case, $N_{kl} = 0$, for $kl \neq CD$. It follows from equations (53)–(55) and (58) that $a_{kl} = b_{kl} = 0$, for $kl \neq CD$ and hence

$$S_{CD}^1 = S_{CD}^2 = 0. \tag{60}$$

Substituting equation (60) into (45) and (46), one obtains the equations governing the solution. These equations are identical in structure to equations (36) and hence the steady state response is a standing wave of the form

$$w = 2\phi_{CD} a_{CD} \cos(\lambda t - \mu_{CD}^a) \cos C\theta + O(\varepsilon). \tag{61}$$

Similar results are obtained for the cases of λ near ω_{NM} and λ near ω_{PQ}.

5.4. THE CASE OF λ NEAR ω_{KL}

In this case $N_{kl} = 0$ for $kl \neq KL$. It follows from equations (55) that $a_{kl} = b_{kl} = 0$, for $kl \neq KL$. A study of equations (53) and (54) reveals that there are two possibilities as follows.

(a) $a_{CD} = a_{NM} = a_{PQ} = 0$. Therefore,

$$S_{KL}^1 = S_{KL}^2 = 0. \tag{62}$$

Substituting equation (62) into (47) and (48), we obtain the equations governing the solution. Again these equations are identical in structure to equations (36): that is, the steady state response is a standing wave of the form

$$w = 2\phi_{KL} a_{KL} \cos(\lambda t - \mu_{KL}^a) \cos K\theta + O(\varepsilon). \tag{63}$$

(b) a_{CD}, a_{NM} and a_{PQ} are non-zero. Here one cannot arrive at the result given by equation (62) and hence one must conclude that the only possible steady state response is a standing wave. The highest mode involved in the internal resonance can appear in the response either as a standing wave (i.e., $a_{KL} = b_{KL}$) of the form $2\phi_{KL}a_{KL}\cos(\lambda t - \mu^a_{KL})\cos K\theta$, or as a traveling wave (i.e., $a_{KL} \neq b_{KL}$) of the form

$$\phi_{KL}[a_{KL}\cos(\lambda t - \mu^a_{KL} + K\theta) + b_{KL}\cos(\lambda t - \mu^b_{KL} - K\theta)].$$

Thus, the steady-state response is described by either (1) a superposition of the standing wave components of all the modes involved in the internal resonance, or (2) a superposition of the standing wave components of all the lower modes (i.e., CD, NM, PQ) and the traveling wave component of the highest mode (i.e., KL) in the internal resonance.

6. SUMMARY

A systematic analysis of the forced response of a clamped circular plate subjected to harmonic excitations is presented. The general problem, including asymmetric vibrations and traveling waves, is a difficult exercise in analysis and the present approach is shown to provide a great deal of clarity and insight into the nature of the non-linear forced resonant response. The effects of an internal resonance involving four modes are evaluated.

The steady state resonant response, in the first approximation, exhibits the following features.

(1) In the absence of internal resonances or when the frequency of excitation is near one of the lower modes involved in the internal resonance, the steady state response can only have the form of a standing wave.

(2) When the frequency of the excitation is near the highest mode involved in the internal resonance, the steady state response is given by one of the following two forms: (a) a superposition of the standing wave components of all the modes involved in the internal resonance, or (b) a superposition of the standing wave components of all the lower modes and the traveling wave component of the highest mode involved in the internal resonance.

Finally, we note that the general results obtained here can be reduced to those of Part I.

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APPENDIX

Coefficients S_j and frequency combinations Λ_j in equations (29)

j	S_j	Λ_j
1	$A_{cd} A_{nm} A_{pq}$	$\omega_{cd} + \omega_{nm} + \omega_{pq}$
2	$A_{cd} A_{nm} \bar{B}_{pq}$	$\omega_{cd} + \omega_{nm} - \omega_{pq}$
3	$A_{cd} \bar{B}_{nm} A_{pq}$	$\omega_{cd} - \omega_{nm} + \omega_{pq}$
4	$\bar{B}_{cd} A_{nm} A_{pq}$	$-\omega_{cd} + \omega_{nm} + \omega_{pq}$
5	$\bar{B}_{cd} \bar{B}_{nm} \bar{B}_{pq}$	$-\omega_{cd} - \omega_{nm} - \omega_{pq}$
6	$\bar{B}_{cd} \bar{B}_{nm} A_{pq}$	$-\omega_{cd} - \omega_{nm} + \omega_{pq}$
7	$\bar{B}_{cd} A_{nm} \bar{B}_{pq}$	$-\omega_{cd} + \omega_{nm} - \omega_{pq}$
8	$A_{cd} \bar{B}_{nm} \bar{B}_{pq}$	$\omega_{cd} - \omega_{nm} - \omega_{pq}$